Polynomial Ornstein-Uhlenbeck volatility models for SPX & VIX: fast pricing and calibration

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Based on the coauthored paper: "Fourier-Laplace transforms in polynomial Ornstein-Uhlenbeck volatility models" with

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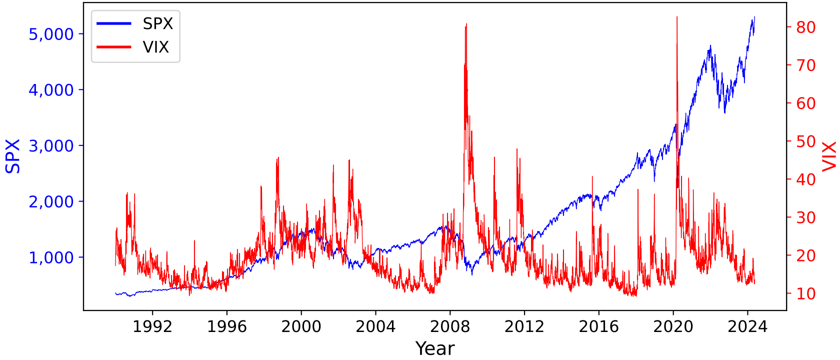
Motivation: the SPX & VIX joint calibration problem

History of SPX & VIX



- **SPX**: Standard and Poor's 500, the benchmark index representing the performance of 500 large U.S. companies.
- **VIX**: the "fear" index that reflects the market's expectation for the volatility of the SPX over the next 30 days.

- **SPX derivatives** have always played a crucial role in financial markets.
- **VIX derivatives** allow **volatility** trading strategies, such as hedging volatility risk, speculating on market movements, and enhancing portfolio diversification.

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- **SPX derivatives** have always played a crucial role in financial markets.
- **VIX derivatives** allow **volatility** trading strategies, such as hedging volatility risk, speculating on market movements, and enhancing portfolio diversification.
- A model that can replicate the observed market prices of SPX and VIX derivatives.
- This process typically involves two steps
  1. **Model:** Find a model with parameters that can be efficiently simulated numerically (for pricing and calibration).
  2. **Calibration:** Adjust the model parameters to minimize the error between model outputs and observed market prices (such as derivative prices or implied volatilities).

Motivation: the SPX & VIX joint calibration problem

- Both SPX and VIX are important. And in practice, VIX is constructed from prices of SPX derivatives.
- The two indexes should be modeled consistently:
  1. **ONE** model to jointly calibrate and fit implied volatility (thus also derivatives price) of SPX and VIX
  2. VIX: expected volatility of the SPX over the next 30 days $\rightarrow$ SPX should be fitted up to 30 days ahead of VIX

This is called the (SPX & VIX) **joint calibration problem**

Stochastic volatility models were introduced around the 1990s and have been widely used in finance (e.g., the Heston model). In the **joint calibration problem,** some unconventional stochastic volatility models have been considered.

Motivation: the SPX & VIX joint calibration problem

unconventional stochastic volatility models for the joint calibration problem

Stochastic volatility models were introduced around the 1990s and have been widely used in finance (e.g., the Heston model). In the **joint calibration problem,** some unconventional stochastic volatility models have been considered.

(I) *Jump diffusion (**discontinuous** spot price)*:

- (Cont and Kokholm, 2013): one of the earliest successful attempts at joint calibration
- (Baldeaux and Badran, 2014): 3/2 model plus jumps

(II) *Multi-factors (volatility depending on **multiple** processes)*:

- (Fouque and Saporito, 2018): Heston model with stochastic vol of vol
- (Rømer, 2022): multi-factor Markovian volatility model
- (Guyon and Lekeufack, 2023): path-dependent volatility

(III) *Rough volatility (**non-Markovian**)*:

- (Gatheral, Jusselin, and Rosenbaum, 2020): quadratic rough Heston
- (Alessandro, Sergio, Scotti, et al., 2024): rough Heston with added Hawkes jumps

Motivation: the SPX & VIX joint calibration problem

- It is **widely believed** (until recently) that *conventional one-factor, continuous Markovian stochastic volatility models* are unable to jointly calibrate the SPX & VIX volatility surfaces.

- Our main motivation is thus to find a model such that:

1. *It is a one-factor continuous Markovian stochastic volatility model, i.e.* ***without*** *using multiple-factors, jumps, roughness.*

2. *It allows* ***fast pricing****, enabling an efficient* ***calibration****.*

3. *It can jointly calibrate and fit SPX & VIX volatility surfaces.*

Motivation: the SPX & VIX joint calibration problem

the Polynomial Ornstein-Uhlenbeck volatility model

Our model: the **Polynomial Ornstein-Uhlenbeck (OU) volatility model**.

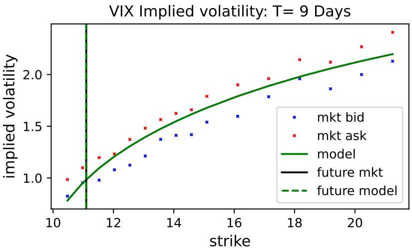
1. *It is a one-factor continuous Markovian stochastic volatility model, i.e. without using multiple-factors, jumps, roughness.*

2. *It allows **fast pricing of** VIX derivatives by integrating against a Gaussian density and **fast pricing** of SPX derivatives by Fourier inversion.*

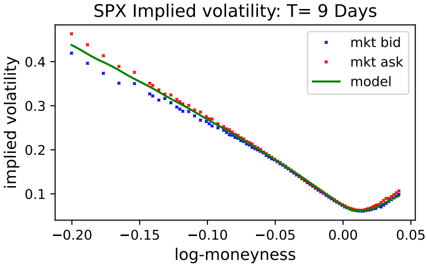
3. *It can jointly calibrate and fit SPX & VIX volatility surfaces.*

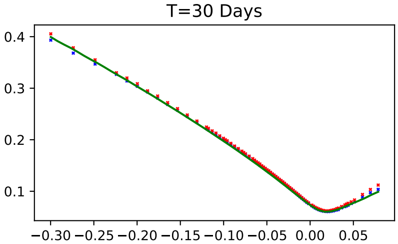
SPX & VIX joint calibration

a quick illustration of the model



Joint calibration of SPX IV, VIX IV, and VIX futures on 23 October 2017 via **Fourier** for SPX and **numerical integration** for VIX. The blue and red dots are market bid/ask, with the green lines are model fit. The vertical bar represents the VIX futures





Polynomial Ornstein-Uhlenbeck volatility models

$$
\begin{aligned}
dS_t &= S_t \sigma_t dB_t, \quad B = \rho W + \sqrt{1-\rho^2} W^{\perp}, \\
\sigma_t &= g_0(t) p(X_t), \quad p(x) = \sum_{k=0}^{\infty} p_k x^k, \\
dX_t &= (a + bX_t)dt + c dW_t.
\end{aligned}
$$

- p is either a polynomial (or an exponential function), X_t is a standard Ornstein-Uhlenbeck process.

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- p is either a polynomial (or an exponential function), X_t is a standard Ornstein-Uhlenbeck process.
- 1. This includes the following models:
  1. Stein-Stein/Schöbel Zhu model: $p(x) = x$,
  2. One-factor Bergomi model: $p(x) = e^x$,
  3. Quintic Ornstein-Uhlenbeck volatility model: $p(x) = p_0 + p_1 x + p_3 x^3 + p_5 x^5$.
- 2. This enables the **fast pricing** of VIX and SPX derivatives used for the **joint calibration**.

Fast pricing of VIX derivatives: Explicit Formula

Fast pricing of VIX derivatives

One major advantage of the Polynomial Ornstein-Uhlenbeck model is the explicit expression of the prices of VIX derivatives:

Fast pricing of VIX derivatives

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An explicit expression for the VIX derivatives price

Given the maturity T, there exists a polynomial $q_T(x)$ such that, for all payoff functions Φ, the VIX derivative price is given by

$$\mathbb{E}\left[\Phi(\mathrm{VIX}_T)\right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi\left(\sqrt{q_T(x)}\right) e^{-x^2/2} dx.$$

Fast pricing of VIX derivatives

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This integral can be computed efficiently using numerical techniques $\rightarrow$ **fast pricing** of VIX derivatives

Fast pricing of SPX derivatives: Fourier Inversion

$$
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dS_t &= S_t\sigma_t dB_t, \quad S_0 > 0, \quad B = \rho W + \sqrt{1-\rho^2}W^{\perp}, \\
\sigma_t &= g_0(t)p(X_t), \quad p(x) = \sum_{k=0}^{\infty} p_k x^k, \\
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- Monte Carlo method: pathwise numerical simulation: $X_t \Rightarrow \sigma_t \Rightarrow S_t$ (=SPX).

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- Monte Carlo method: pathwise numerical simulation: $X_t \Rightarrow \sigma_t \Rightarrow S_t$ (=SPX).
- Fourier method: **fast pricing** by computing the characteristic function and then applying Fourier inversion to obtain a pricing formula.

$$dS_t = S_t \sigma_t dB_t, \quad S_0 > 0, \quad B = \rho W + \sqrt{1-\rho^2} W^{\perp},$$

$$\sigma_t = g_0(t) p(X_t), \quad p(x) = x,$$

$$dX_t = (a + bX_t)dt + c dW_t.$$

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- The Stein-Stein model is affine in $(1, X, X^2)$ such that the characteristic function of $\log S$ is given by

$$\mathbb{E}\Big[\exp\Big(iu\log(\frac{S_T}{S_t})\Big)|\mathcal{F}_t\Big] = \exp\left(\psi_0(t) + \psi_1(t)X_t + \psi_2(t)X_t^2\right), \quad t \le T,$$

where $u \in \mathbb{R}$, (ψ_0, ψ_1, ψ_2) is the solution to a standard system of finite dimensional Riccati equations $\to$ **existence** and **uniqueness** are well-known

$$
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dS_t &= S_t \sigma_t dB_t, \quad S_0 > 0, \quad B = \rho W + \sqrt{1-\rho^2} W^{\perp}, \\
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dX_t &= (a + bX_t)dt + c dW_t.
\end{aligned}
$$

- For a general p, we expect the model to be affine in $(1, X, \cdots, X^n, \cdots)$ with the Ansatz for the characteristic function:

$$
\mathbb{E}\Big[\exp\Big(iu \log(\frac{S_T}{S_t}) \Big) | \mathcal{F}_t \Big] = \exp\Big(\sum_{k \geq 0} \psi_k(T-t) X_t^k \Big), \quad t \leq T.
$$

- Similar expansions of the characteristic function as a power series have also appeared in recent works of (Cuchiero, Svaluto-Ferro, and Teichmann, 2023; Friz, Gatheral, and Radoičić, 2022; Abi Jaber and Gérard, 2024).

Characteristic function and infinite-dimensional Riccati equations

$$\mathbb{E}\Big[\exp\Big(iu\log(\frac{S_T}{S_t})\Big)|\mathcal{F}_t] = \exp\Big(\sum_{k\geq 0}\psi_k(T-t)X_t^k\Big). \qquad (1)$$

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The (infinite dimensional) vector $\psi(t)$ solves a system of Ricatti equations:

$$\psi'(t) = P(t) + Q(t)\psi(t) + K\psi(t) * K\psi(t). \tag{2}$$

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- We establish theoretical results to build (1) and numerical scheme to solve infinite dimensional Riccati equations (2).

Characteristic function and infinite-dimensional Riccati equations

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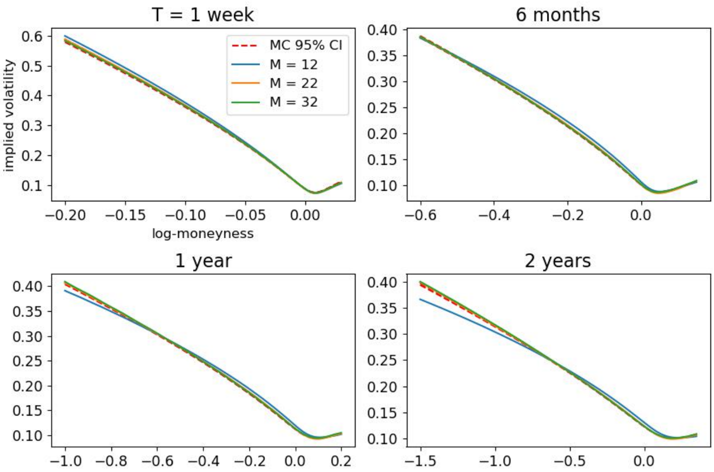
- We establish theoretical results to build (1) and numerical scheme to solve infinite dimensional Riccati equations (2).
- Solve Ricatti equations (2) $\rightarrow$ Compute characteristic function by (1) $\rightarrow$ Pricing formula based on **Fourier inversion** $\rightarrow$ **Fast pricing** of SPX derivatives.

$$dS_t = S_t\sigma_t dB_t, \quad S_0 > 0, \quad B = \rho W + \sqrt{1-\rho^2} W^{\perp},$$
$$\sigma_t = g_0(t)p(X_t), \quad p(x) = p_0 + p_1 x + p_3 x^3 + p_5 x^5, \quad p_0, p_1, p_3, p_5 \geq 0,$$
$$dX_t = \alpha\epsilon^{-1} X_t dt + \epsilon^{\alpha} dW_t.$$

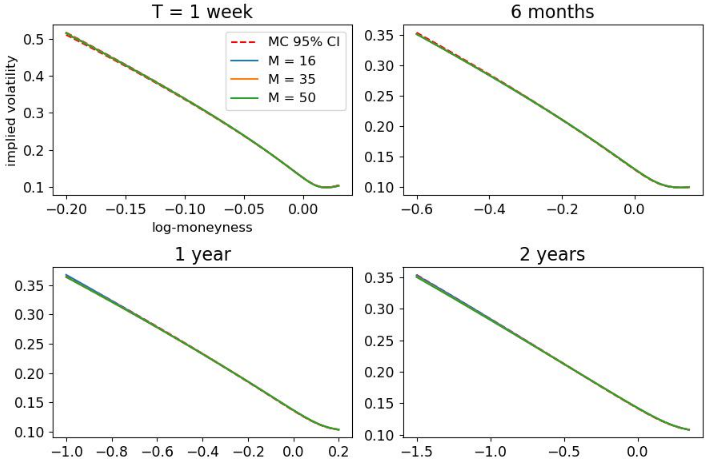
- We choose the following parameters: $\rho = -0.65$, $(p_0, p_1, p_3, p_5) = (0.01, 1, 0.214, 0.227)$, $\xi_0(t) = 0.025$, $g_0(t) = \sqrt{\frac{\xi_0(t)}{\mathbb{E}[p(X_t)^2]}}$, $\alpha = -0.6$, $\varepsilon = 1/52$, which are typical values one can expect from calibrating the model to SPX & VIX volatility surfaces.



SPX implied volatility of different maturities in the quintic Ornstein-Uhlenbeck model, computed with our algorithm with different level M. Dotted red lines are Monte-Carlo 95% interval computed with 500,000 simulations and $n = 10,000$ number of steps per maturity slice.

$$dS_t = S_t \sigma_t dB_t, \quad S_0 > 0, \quad B = \rho W + \sqrt{1-\rho^2} W^{\perp},$$
$$\sigma_t = g_0(t) p(X_t), p(x) = \exp(\frac{1}{2}\eta x),$$
$$dX_t = \alpha \epsilon^{-1} X_t dt + \epsilon^{\alpha} dW_t, g_0(t) = \sqrt{\xi_0(t)} \exp\left(-\frac{1}{4}\eta^2 \mathbb{E}(X_t^2)\right).$$

- We choose the parameters: $\rho = -0.7, \varepsilon = 1/52, \alpha = -0.7, \eta = 1.2, \xi_0(t) = 0.025$ for the numerical experiment, which are typical values one can expect from calibrating the model to SPX volatility surface.



SPX implied volatility of different maturities in the one-factor Bergomi model, computed with our algorithm with different level M. Dotted red lines are Monte-Carlo 95% interval computed with 500,000 simulations and $n = 10,000$ number of steps per maturity slice.

SPX & VIX: from fast pricing to joint calibration

Now, we are able to perform a (fast) **joint calibration** based on our choice of **Polynomial Ornstein-Uhlenbeck volatility models** and **fast pricing** method.

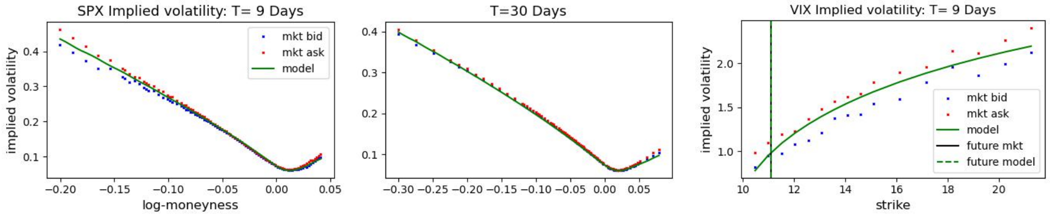
- Step 1: We fix the model by choosing a set of free parameters Θ to calibrate.
- Step 2: Given a set of parameters Θ, we are able to do the **fast pricing** of SPX & VIX derivatives prices and implied volatility.
- Step 3: We fix an error function $\mathbf{R}$, which evaluates the difference between the model and the real market data by the derivatives prices and implied volatility.

  The **joint calibration** is performed by using **fast pricing** to find parameters Θ to minimize the error $\mathbf{R}$.

In the Quintic Ornstein-Uhlenbeck volatility models, we choose $\Theta := \{p_0, p_1, p_3, p_5, \rho, \alpha\}$, and the error function $\mathbf{R}$ as the sum of root mean squared error (RMSE) of future price and implied volatility.

In the Quintic Ornstein-Uhlenbeck volatility models, we choose $\Theta := \{p_0, p_1, p_3, p_5, \rho, \alpha\}$, and the error function $\mathbf{R}$ as the sum of root mean squared error (RMSE) of future price and implied volatility.

Then the calibrated parameters are $(p_0, p_1, p_3, p_5) = (0.0202, 1.3332, 0.0578, 0.0071)$, $\rho = -0.6763, \alpha = -0.6821$. And the numerical illustration for the calibrated parameters:



Quintic Ornstein-Uhlenbeck model (green lines) jointly calibrated to the SPX and VIX smiles (bid/ask in blue/red) on 23 October 2017 via Fourier using the Nelder-Mead optimization algorithm. The truncation level of the Riccati solver is set at $M = 32$.

Conclusion

We introduced **Polynomial Ornstein-Uhlenbeck volatility model**, which enables SPX & VIX **fast pricing** and **joint calibration**, thereby supporting hedging, trading and enhancing portfolio diversification from both the spot price (SPX) and volatility (VIX) perspectives.

- It is a conventional one-factor continuous Markovian stochastic volatility model.
- It allows fast **joint calibration** by
  1. **fast pricing** of VIX derivatives: integrating an explicit formula against Gaussian density
  2. **fast pricing** of SPX derivatives: solving Riccati equations which is used to compute the characteristic function, then applying the Fourier inversion
- It can jointly calibrate and fit SPX & VIX volatility surfaces.

$$\mathrm{VIX}_T^2 = \frac{100^2}{\Delta} \sum_{i=0}^{2deg(p)} \beta_i(T) X_T^i = q(X_T),$$

with $(p * p)_k = \sum_{j=0}^{k} p_j p_{k-j}$ the discrete convolution, $\binom{k}{i}$ the binomial coefficient, and

$$\beta_i(T) = \sum_{k=i}^{2deg(p)} (p * p)_k \binom{k}{i} \int_0^{\Delta} g_0^2(T+t) e^{bkt} \mathbb{E}\left[Y_t^{k-i}\right] dt,$$

where $Y_t = a \int_0^t e^{-bu} du + c \int_0^t e^{-bu} dW_u$ is Gaussian with explicit moments.

$$\mathbb{E}\left[\Phi(\mathrm{VIX}_T)\right] = \mathbb{E}\left[\Phi\left(\sqrt{q(X_T)}\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi\left(\sqrt{q(\sigma_{X_T} x + m_{X_T})}\right) e^{-x^2/2} dx.$$

with m_{X_T} being the expectation of X_T, $\sigma_{X_T}^2$ being the variance of X_T.

$$\mathbb{E}\left[\exp\left(iu\log(\frac{S_T}{S_t})\right)|\mathcal{F}_t\right] = \exp\left(\sum_{k\geq 0}\psi_k(T-t)X_t^k\right). \tag{3}$$

$$\mathbb{E}\Big[\exp\Big(iu\log(\frac{S_T}{S_t})\Big)|\mathcal{F}_t\Big] = \exp\Big(\sum_{k\geq 0}\psi_k(T-t)X_t^k\Big). \tag{3}$$

$$\begin{aligned}\psi_k'(t) &= \frac{1}{2}(-u^2-iu)g_0^2(T-t)(p*p)_k + bk\psi_k(t) + a(k+1)\psi_{k+1}(t)\\ &+ \frac{c^2(k+2)(k+1)}{2}\psi_{k+2}(t) + iuc\rho g_0(T-t)(p*\widetilde{\psi}(t))_k\\ &+ \frac{c^2}{2}(\widetilde{\psi}(t)*\widetilde{\psi}(t))_k, \quad \psi_k(0)=0, \quad \widetilde{\psi}_k = (k+1)\psi_{k+1}, k\in\mathbb{N}.\end{aligned} \tag{4}$$

Given a maturity T and a strike K, we aim to compute the call option price at time t from the characteristic function.

We adopt the pricing formula suggested from Lewis (2001):

$$C_t(S_t, K, T) := \mathbb{E}\left[(S_T - K)^+ | \mathcal{F}_t\right] = S_t - \frac{K}{\pi} \int_0^\infty \mathfrak{Re}\left[e^{\left(iu+\frac{1}{2}\right)k_t} \varphi\left(u - \frac{i}{2}\right)\right] \frac{du}{u^2 + \frac{1}{4}}, \tag{5}$$

where $k_t := \log(K/S_t)$ is the log-moneyness and $\varphi(u) = \mathbb{E}\Big[\exp\Big(iu\log(\frac{S_T}{S_t})\Big)|\mathcal{F}_t\Big]$ is the characteristic function.

Take Quintic Ornstein-Uhlenbeck volatility model for example: we fix $\varepsilon = 1/52$, and the forward variance curve $\xi_0(t)$ comes directly from market data. There are **six** model parameters:

$$\Theta := \{p_0, p_1, p_3, p_5, \rho, \alpha\}$$

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To calibrate our model, we solve the optimisation problem involving sum of root mean squared error (RMSE):

$$\min_{\Theta} \left\{ c_1 \sqrt{\sum_{i,j} \left(\sigma^{\Theta}_{\text{spx}}(T_i, K_j) - \sigma^{\text{mkt}}_{\text{spx}}(T_i, K_j)\right)^2} + c_2 \sqrt{\sum_{i,j} \left(\sigma^{\Theta}_{\text{vix}}(T_i, K_j) - \sigma^{\text{mkt}}_{\text{vix}}(T_i, K_j)\right)^2} \right. \tag{6}$$

$$\left. + c_3 \sqrt{\sum_{i} \left(F^{\Theta}_{\text{vix}}(T_i) - F^{\text{mkt}}_{\text{vix}}(T_i)\right)^2} \right\}. \tag{7}$$

T_i : maturity, K_j : strike, σ : implied volatility, F : future price, mkt: real market, Θ : our model, c_1, c_2, and c_3: positive weights.

Wrap-up
discussions

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